Comments on the problem

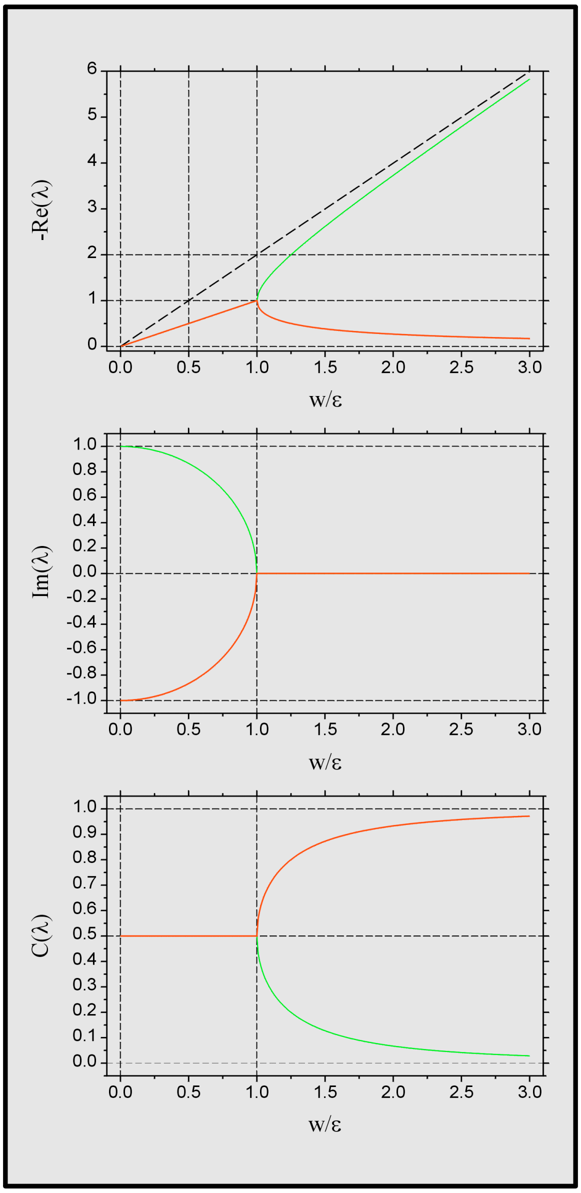
The relaxation operator has the form:

$$\mathbf{R} = \begin{pmatrix} -\mathrm{w} - i\varepsilon & \mathrm{w} \\ \mathrm{w} & -\mathrm{w} + i\varepsilon \end{pmatrix}.$$

It describes relaxation between two isomer shift states having equal probability to occur. States are separated by the energy $2\varepsilon > 0$, and the relaxation occurs at the average rate of $\mathrm{w} \geq 0$. System remains at equilibrium with the energy origin set in the midpoint of the respective shifts. The eigensolutions [$\det(\mathbf{R} - \boldsymbol{\lambda}) = 0$, $\mathbf{VRU} = \boldsymbol{\lambda}$ with $\mathbf{VU} = \mathbf{UV} = \mathbf{1}$ and $\boldsymbol{\lambda}$ being diagonal complex matrix] are as follows:

$$\lambda = -\mathrm{w} \pm i\sqrt{\varepsilon^2 - \mathrm{w}^2} : \mathrm{C}(\lambda) = \tfrac{1}{2} \text{ for} : 0 \leq \mathrm{w} \leq \varepsilon \text{ and} :$$

$$\lambda = -\mathrm{w} \pm \sqrt{\mathrm{w}^2 - \varepsilon^2} : \mathrm{C}(\lambda) = \tfrac{1}{2}\left[1 \pm \frac{\sqrt{\mathrm{w}^2 - \varepsilon^2}}{\mathrm{w}}\right] \text{ for} : \mathrm{w} \geq \varepsilon .$$



Note that both solutions merge for $\mathrm{w} = \varepsilon$. Eigenvalues λ and relative intensities $\mathrm{C}(\lambda)$ are plotted (at the side) versus w/ε. Relative intensities are calculated here as $\mathrm{C}(\lambda) = b_\lambda / \sum_{\lambda'} b_{\lambda'}$ with:

$$b_\lambda = \sqrt{\left(\sum_{jk} V_{\lambda j} U_{k\lambda}\right)\left(\sum_{jk} V_{\lambda j} U_{k\lambda}\right)^*} .$$

This is typical example of the so-called motional narrowing process.

How to proceed

The first step is to find eigenvalues applying equation $\det(\mathbf{R} - \boldsymbol{\lambda}) = 0$. There are two complex eigenvalues, i.e., four parameters to find. There are separate solutions in the slow relaxation limit ($0 \leq \mathrm{w} \leq \varepsilon$) and fast relaxation limit ($\mathrm{w} \geq \varepsilon$). Solutions are obtained as solutions of the following equation in the complex domain:

$$(-\mathrm{w} - i\varepsilon - \lambda)(-\mathrm{w} + i\varepsilon - \lambda) - \mathrm{w}^2 = 0 .$$

The matrix $\mathbf{U}$ contains four complex elements (eigenvectors are columns), i.e., eight parameters to establish. The same statement applies to the matrix $\mathbf{V}$ with eigenvectors as rows. Hence, one has to find sixteen parameters separately in the slow and fast relaxation limits. The equation $\mathbf{VRU} = \boldsymbol{\lambda}$ yields eight equations for real parameters, while the equation $\mathbf{VU} = \mathbf{1}$ yields another eight equations of the same type (the symbol $\mathbf{1}$ stands for the unit matrix). Hence, one can find all elements of the left ($\mathbf{V}$) and right ($\mathbf{U}$) eigenvectors and calculate respective relative line intensities in both relaxation limits. Note that the parameters $C(\lambda)$ have to satisfy the following relationships for any physically meaningful value of the argument, i.e., for $w/\varepsilon \geq 0$: $C(\lambda) \geq 0$ and $\sum_{\lambda} C(\lambda) = 1$.

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